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## The $k = 2$ string tension in four dimensional SU(N) gauge theories

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### **Abstract.**

We calculate the  $k = 2$  string tension in SU(4) and SU(5) gauge theories in 3+1 dimensions, and compare it to the  $k = 1$  fundamental string tension. We find, from the continuum extrapolation of our lattice calculations, that  $\sigma_{k=2}/\sigma_f = 1.40 \pm 0.08$  in the SU(4) gauge theory, and that  $\sigma_{k=2}/\sigma_f = 1.56 \pm 0.10$  in SU(5). We remark upon the way this might constrain the dynamics of confinement and the intriguing implications it might have for the mass spectrum of SU( $N$ ) gauge theories. We also note that these results agree closely with the MQCD-inspired conjecture that the SU( $N$ ) string tension varies as  $\sigma_k \propto \sin(\pi k/N)$ .

# 1 Introduction

It is well known that confinement in four dimensional  $SU(N)$  gauge theories leads to a linear potential between static test charges in the fundamental representation, so that there is a non-zero fundamental string tension  $\sigma_f \neq 0$ . For  $SU(N \geq 4)$  there is, however, the possibility of new stable strings which join test charges in representations higher than the fundamental. One can conveniently label these by the way the test charge transforms under a gauge transformation,  $z$ , that belongs to the centre,  $Z_N$ , of the group. If the charge acquires a factor  $z^k$  we shall refer to the string as having  $N$ -ality  $k$ , and the corresponding string tension will be denoted by  $\sigma_k$ . So the fundamental string has  $N$ -ality  $k = 1$  and we shall refer to it as  $\sigma_f$  or  $\sigma_{k=1}$ , interchangeably.

Because gluons transform trivially under the centre, such a  $k$ -string will not be screened down to a  $k'$ -string, if  $k' \neq k$ . If, however,  $z^k = z^{k'} \forall z$  then the  $k$ -string can transform into a  $k'$ -string and so one trivially has that  $\sigma_k = \sigma_{k'}$ . For example, in the case of  $SU(3)$  a  $k = 2$  string joining a diquark source to a distant anti-diquark source will be simply the usual fundamental  $k = 1$  string over almost all of the separation. (In  $SU(3)$  three fundamental strings can form a colour singlet.) We have to go to at least  $SU(4)$  in order to have the possibility of a genuinely different  $k = 2$  string, and to  $SU(6)$  for a  $k = 3$  string.

One can form a  $k$ -string by simply using static test charges that consist of  $k$  fundamental charges. So the most trivial possibility is that a ' $k$ -string' is nothing more than  $k$  separate fundamental strings joining the two test charges. In this case we will have  $\sigma_k = k\sigma_f$ . So for example, if  $\sigma_{k=2} \not\leq 2\sigma_f$  then the  $k = 2$  string is just like two  $k = 1$  strings and we have nothing new. There is however a non-trivial possibility. If  $\sigma_{k=2} < 2\sigma_f$  then the  $k = 2$  string will be stable and will constitute a new kind of string. We may regard it as a bound state of two  $k = 1$  strings. Similar comments apply to higher values of  $k$ .

As emphasised in [1], one reason why the lattice calculation of  $\sigma_k$  in  $SU(N)$  gauge theories is particularly interesting is that it tests the conjecture [1] that the ratio  $\sigma_k/\sigma_f$  has a universal value in 'QCD-like' theories:

$$\frac{\sigma_k}{\sigma_f} = \frac{\sin \frac{k\pi}{N}}{\sin \frac{\pi}{N}}. \quad (1)$$

This conjecture is based on calculations in brane (M-)theory [1]. The resulting variants of QCD go under the generic name of MQCD. One finds that in MQCD with  $\mathcal{N} = 1$  supersymmetry [2, 3], the string tensions of  $k$ -strings satisfy eqn(1). The same is true for  $\mathcal{N} = 2$ , for  $\mathcal{N} = 2$  softly broken to  $\mathcal{N} = 1$  and for non-supersymmetric MQCD [1]. This is so even though the actual values of  $\sigma_k$  differ in these theories. Hence the conjecture [1] that the ratio of string tensions is 'universal' in QCD-like theories and takes the value in eqn(1).

Another reason for being interested in the value of  $\sigma_k$  is that the existence of  $k$ -strings may have a striking impact on the mass spectrum of  $SU(N)$  gauge theories

in the large- $N$  limit [4]. A natural model for the glueball states that make up the spectrum of pure gauge theories is as quantised closed strings of fundamental flux. Such a picture is most compelling for highly excited states which are large compared to the intrinsic width of the flux tube, but may be a reasonably accurate representation for the low-lying spectrum as well. A simple and explicit model based on this idea was suggested some time ago [5] and comparisons [6, 7, 8] with the detailed  $SU(N)$  lattice mass spectra available in 2+1 dimensions [9] show that it works reasonably well. As emphasised in [8] one can build towers of glueball states on any stable  $k$ -string and these will be identical up to a rescaling of the overall mass scale by a factor of  $\sqrt{\sigma_k/\sigma_f}$  (if one neglects the finite width of the flux tube). This argument neglects mixing and decay, but this becomes appropriate as  $N \rightarrow \infty$ . Such a mass spectrum would provide remarkable evidence for the underlying string structure of the glueball states. All this does however depend on the existence of stable  $k$ -strings and the details of the  $SU(N)$  spectrum will depend on the actual values of the corresponding string tensions.

Finally we remark that knowing the value of  $\sigma_k/\sigma_f$  will constrain the details of any proposed confinement mechanism.

In this paper we will present some results on the  $k = 2$  string tension and will compare it to the fundamental ( $k = 1$ ) tension. We do this for both  $SU(4)$  and  $SU(5)$ , although our calculations will be more precise in the former case. There exist previous calculations [10] of the lattice  $SU(4)$   $k = 2$  string tension which show that  $\sigma_k < 2\sigma_f$ . However these calculations were performed at quite a high temperature (although still in the confining phase) and were not accurate enough to make a useful statement about the continuum limit. We intend to perform calculations that are accurate enough, in the continuum limit, to provide a significant test of the MQCD conjecture in eqn(1). These calculations are part of our ongoing study [11] of the physical properties of D=3+1  $SU(N)$  gauge theories, and a similar calculation is in progress elsewhere [12].

## 2 Calculations

To calculate  $\sigma_k$  we first construct a  $k$ -string that closes upon itself through a spatial boundary (so no explicit static test charges are needed). We then calculate the correlation function of two such  $k$ -strings as a function of their (Euclidean) time separation, using standard Monte Carlo simulation methods. At sufficiently large separation  $t$ , the correlation function will fall as a simple exponential  $\propto \exp(-m_k t)$  where  $m_k$  is the mass of the lightest  $k$ -string state. Linear confinement means that this mass will be linear in the length of the string, which we denote by  $l \equiv aL$  where  $a$  is the lattice spacing and  $L$  is the lattice spatial size in lattice units. In addition to this linear dependence the first correction is universal [13] and we assume that the universality class is that of a simple bosonic string. (There is evidence from many previous  $SU(2)$  and  $SU(3)$  lattice calculations that points to this.) Using the string correction appropriate to our

closed periodic string [14] we therefore have

$$am_k = a^2\sigma_k L - \frac{\pi}{3L} + \dots \quad (2)$$

where everything has been expressed in lattice units. So once we have calculated the mass  $am_k$  we can use eqn(2) to extract a value for  $a^2\sigma_k$ .

In practice a calculation using the simplest lattice  $k$ -string operator is inefficient because the overlap onto the lightest string state is small and so one has to go to large values of  $t$  before the contribution of excited states has died away; and there the signal disappears into the statistical noise. There are standard methods [15] for curing this problem, using blocked (smeared) link operators and variational techniques. This is described in detail in, for example, [9] for the case of a  $k = 1$  string. Let  $\text{Tr } l_1$  be such a string, so that  $l_1$  is the string before one closes it by taking the trace. Then we can form trial  $k = 2$  strings from  $(\text{Tr } l_1)^2$  and  $\text{Tr } (l_1 l_1)$ . In the variational calculation we allow arbitrary linear combinations of these in the search for the lightest state. (In practice we expect this to belong to the appropriate representation of  $\text{SU}(N)$ .) Details of this, and of our choice of Monte Carlo algorithm, will be given elsewhere [11].

Our  $\text{SU}(4)$  calculations are summarised in Table 1. They span a range of couplings that corresponds roughly, to the range  $\beta \in [5.7, 6.0]$  in the case of  $\text{SU}(3)$  [17]. This is well beyond the bulk phase transition due to the non-trivial phase structure in the fundamental-adjoint coupling plane [16]. This phase structure will be more complex as  $N$  increases, since more couplings, appropriate to representations other than just the fundamental and adjoint, become relevant, but its effects are expected to remain confined to the region between weak and strong coupling. (The phase structure is believed to be due to the differing sensitivity of plaquettes in different representations to the presence of  $Z_k$  monopoles of various  $k$ , and there are more possibilities as  $N$  increases.)

The volumes used in these calculations have been chosen so that if we were working in  $\text{SU}(3)$  on the same volumes (when expressed in physical units e.g. using  $l_\sigma \equiv 1/\sqrt{\sigma_f}$ ) then eqn(2) would hold very accurately. This comparison assumes of course that all these gauge theories are similar – which would be true if they were all quite close to  $\text{SU}(\infty)$ . However we should not assume that this is so at this stage and it would be reassuring to have some explicit evidence for the assumption of linear confinement, and the universal Luscher correction, for  $\text{SU}(4)$  and  $\text{SU}(5)$ . Accordingly we show in Fig.1 how the flux loop mass varies with its length in the  $\text{SU}(4)$  lattice gauge theory, for  $\beta = 10.70$ . We see some explicit evidence that for  $L \geq 10$  eqn(2) is indeed valid, for both  $k = 1$  and  $k = 2$  flux strings. For smaller  $L$  the higher order corrections to this relation are significant, especially so for the  $k = 2$  string whose string tension appears to approach the fundamental string tension as its length shortens. We have performed a separate calculation (although only for  $k = 1$  strings) at the smaller lattice spacing corresponding to  $\beta = 10.9$ , and again, as we see in Fig.2, the relation in eqn(2) holds well for the volumes used in Table 1.

With the standard plaquette action that we are using, the leading lattice correction to the continuum limit is  $O(a^2)$ , and so for sufficiently small  $a$  we can extrapolate to the continuum using

$$\frac{\sigma_k(a)}{\sigma_f(a)} = \frac{\sigma_k(0)}{\sigma_f(0)} + ca^2\sigma_f. \quad (3)$$

In Fig.3 we plot our calculated values of  $a^2\sigma_{k=2}/a^2\sigma_f \equiv \sigma_{k=2}/\sigma_f$  against the simultaneously calculated value of  $a^2\sigma_f$ . On such a plot the continuum extrapolation in eqn(3) is a simple straight line. It is clear that all our points are consistent with lying on such a line and we show the best fit in the plot. From this we extract our continuum limit for the ratio of string tensions:

$$\lim_{a \rightarrow 0} \frac{\sigma_{k=2}}{\sigma_f} = \begin{cases} 1.354 \pm 0.070 & \beta \geq 10.55 \\ 1.405 \pm 0.080 & \beta \geq 10.70 \end{cases} : \text{SU}(4). \quad (4)$$

Both of the fits in eqn(4) are quite good, with confidence levels  $\sim 40\%$ . (The error limits correspond to confidence levels  $\sim 10\%$ .) The  $\beta \geq 10.70$  fit has the virtue of avoiding the  $\beta = 10.55$  calculation which might be biased by the nearby bulk phase transition, but suffers from a shorter lever arm.

Our SU(5) calculations, which are more limited in scope, are summarised in Table 2 and the results for  $\sigma_{k=2}/\sigma_f$  are plotted in Fig.3. The continuum extrapolation, using eqn(3), is clearly less constrained than for SU(4) and leads to:

$$\lim_{a \rightarrow 0} \frac{\sigma_{k=2}}{\sigma_f} = 1.556 \pm 0.100 : \text{SU}(5). \quad (5)$$

In addition we have deliberately tuned the SU(5) couplings so that each of the three values of  $a$  roughly coincides, when expressed in physical units (of  $1/\sqrt{\sigma_f}$ ), with one of the SU(4) couplings. It is plausible that the lattice corrections largely cancel in the relative SU(4) and SU(5) ratios, so that the average

$$\frac{(\sigma_{k=2}/\sigma_f)_{\text{SU}(5)}}{(\sigma_{k=2}/\sigma_f)_{\text{SU}(4)}} = 1.10 \pm 0.03 : 0.38 \geq a\sqrt{\sigma_f} \geq 0.24 \quad (6)$$

should be close to its continuum value.

### 3 Discussion

According to the MQCD conjecture in eqn(1) we should expect

$$\frac{\sigma_{k=2}}{\sigma_f} = \frac{\sin \frac{2\pi}{N}}{\sin \frac{\pi}{N}} = 2 \cos \frac{\pi}{N} = \begin{cases} 0 & \text{SU}(2) \\ 1 & \text{SU}(3) \\ 1.41... & \text{SU}(4) \\ 1.61... & \text{SU}(5) \end{cases} \quad (7)$$

This is plausible for SU(2), where two fundamental charges can form an unconfined colour singlet, and for SU(3), where two fundamental strings can couple to a single fundamental (anti-)string. (We say ‘plausible’ because one could imagine the  $N \geq 4$  string bound states continuing down to unstable, resonant string states for  $N = 3$  and  $N = 2$ .) Our SU(4) results are consistent with eqn(7), within fairly small errors. So are our SU(5) results; although the continuum extrapolation is less accurate. The ratio in eqn(6) is indeed close to the MQCD prediction of  $\simeq 1.14$ . All this appears to provide significant support for the approximate validity of the MQCD conjecture.

To gauge how significant this apparent agreement is, it would be useful to compare with something else. Since our numerical calculations are on a Euclidean lattice with a plaquette action, one obvious comparison is with the strong coupling prediction. This is well-known to be

$$\frac{\sigma_k}{\sigma_f} = \min\{k, N - k\} \stackrel{k=2}{=} \begin{cases} 0 & \text{SU(2)} \\ 1 & \text{SU(3)} \\ 2 & \text{SU(N} \geq 4) \end{cases} \quad (8)$$

which starts out well but is quite inconsistent with our SU(4) and SU(5) calculations. A more intriguing comparison, emphasised in [1], is with the Hamiltonian strong coupling result

$$\frac{\sigma_k}{\sigma_f} = \frac{k(N - k)}{N - 1} \stackrel{k=2}{=} \begin{cases} 0 & \text{SU(2)} \\ 1 & \text{SU(3)} \\ 1.3\bar{3} & \text{SU(4)} \\ 1.50 & \text{SU(5)} \end{cases} \quad (9)$$

This is clearly compatible with our numerical results. Of course one might question the relevance of comparing Euclidean lattice calculations with the strong coupling limit of the Hamiltonian lattice approach. After all the strong coupling limit is not universal and we could find almost anything we wanted in some strong coupling limit or other. (For example, if we were to construct a lattice action that uses link matrices in matrix representations other than just the fundamental.) On the other hand the origin of the formula in eqn(9) has a nice simple physical interpretation that might arise in a more general context, and one might argue that colour-electric flux is better represented with time continuous. It is therefore particularly interesting to note that our errors are already comparable to the difference between the MQCD and Hamiltonian strong coupling predictions. A calculation that is better than ours by, say, a factor of  $O(10)$  would readily discriminate between eqn(8) and eqn(9). Such a calculation is quite feasible and we intend to carry it out in the near future.

In any case, we have shown that bound  $k = 2$  strings states do exist for SU( $N \geq 4$ ) and that the binding is strong for SU(4) and SU(5). This will have implications for the confinement dynamics. For example <sup>1</sup>, in the picture where confinement in SU( $N$ ) gauge theories is driven by the condensation of centre vortices [18] the value of  $\sigma_{k=2}/\sigma_f$

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<sup>1</sup>this argument arose in a discussion with C. Korthals Altes

provides information on the relative density of  $k = 2$  and  $k = 1$  vortices. If one neglects all correlations between vortices, then the calculation of how vortices disorder Wilson loops involves only a simple Poisson distribution [19] and one readily finds that the number of  $k = 2$  vortices piercing unit area (of the minimal surface spanning a Wilson loop) is related to the number of  $k = 1$  vortices by

$$\frac{N_{k=2}^v}{N_{k=1}^v} = \frac{\sigma_{k=1}}{\sigma_{k=2}} - \frac{1}{2} \quad (10)$$

in the case of  $SU(4)$ . So using our calculated string tensions we obtain a constraint on the relative density of doubly charged centre vortices condensed in the  $SU(4)$  vacuum:  $N_2^v/N_1^v \simeq 0.25(5)$ . It is interesting to note that at high temperatures one can calculate the dual 't Hooft string tension in perturbation theory, and one finds [20] a formula that happens to be identical to the Hamiltonian strong coupling result given in eqn(9).

The fact that the  $k = 2$  string tension is much less than twice the fundamental string tension, for  $N = 4$  and  $N = 5$ , suggests the possibility that the mass spectra of these gauge theories might contain prominent states based upon closed loops of such  $k = 2$  strings, that are additional to any states based on closed loops of fundamental flux. It would clearly be interesting to perform lattice calculations of  $SU(N)$  gauge theories that are designed to explore this intriguing possibility.

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SU(4)				
$\beta$	lattice	MC sweeps	$a\sqrt{\sigma_f}$	$\sigma_{k=2}/\sigma_f$
10.55	$8^4$	$2 \times 10^5$	0.3726(30)	1.432(35)
10.70	$10^4$	$10^5$	0.3070(12)	1.380(19)
10.70	$12^4$	$10^5$	0.3058(12)	1.377(35)
10.90	$12^4$	$10^5$	0.2421(17)	1.358(42)
11.10	$16^4$	$9.6 \times 10^4$	0.2022(11)	1.413(42)

Table 1: The ratio of  $k = 2$  and  $k = 1$  string tensions in SU(4) from lattice calculations with various lattice spacings (given in units of the fundamental string tension).

SU(5)				
$\beta$	lattice	MC sweeps	$a\sqrt{\sigma_f}$	$\sigma_{k=2}/\sigma_f$
16.755	$8^4$	$10^5$	0.3845(20)	1.54(6)
16.975	$10^4$	$1.6 \times 10^5$	0.3027(21)	1.49(7)
17.270	$12^4$	$8 \times 10^4$	0.2480(11)	1.56(5)

Table 2: The ratio of  $k = 2$  and  $k = 1$  string tensions in SU(5) for lattice calculations with various lattice spacings (given in units of the fundamental string tension).

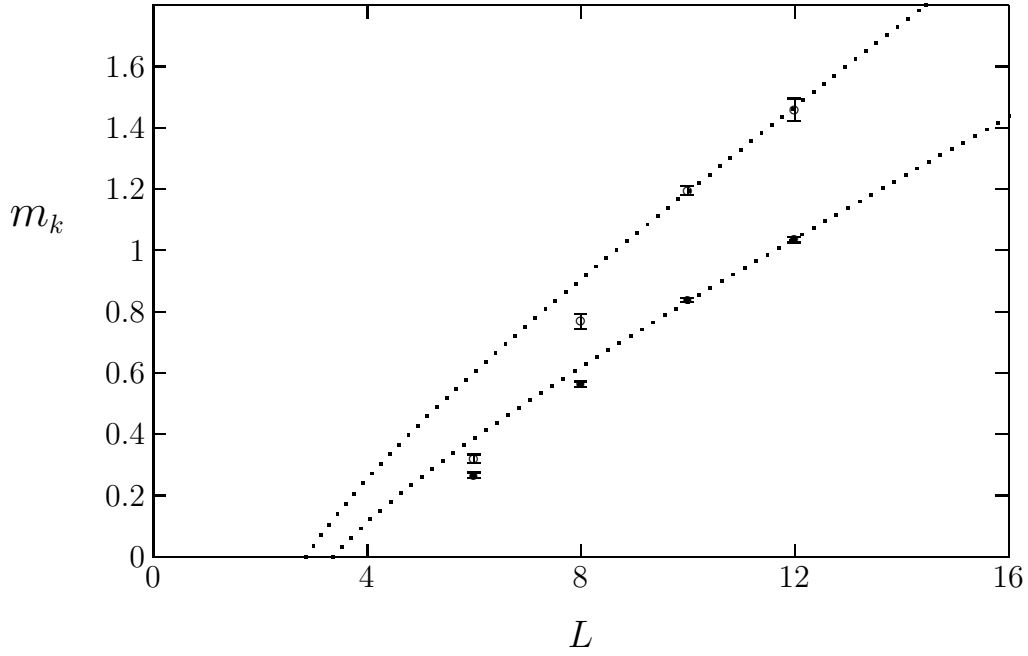


Figure 1: The masses of the fundamental,  $\bullet$ , and  $k = 2$ ,  $\circ$ , flux loops versus their length, at  $\beta = 10.7$  for SU(4). Shown also is the variation one expects from eqn(2).

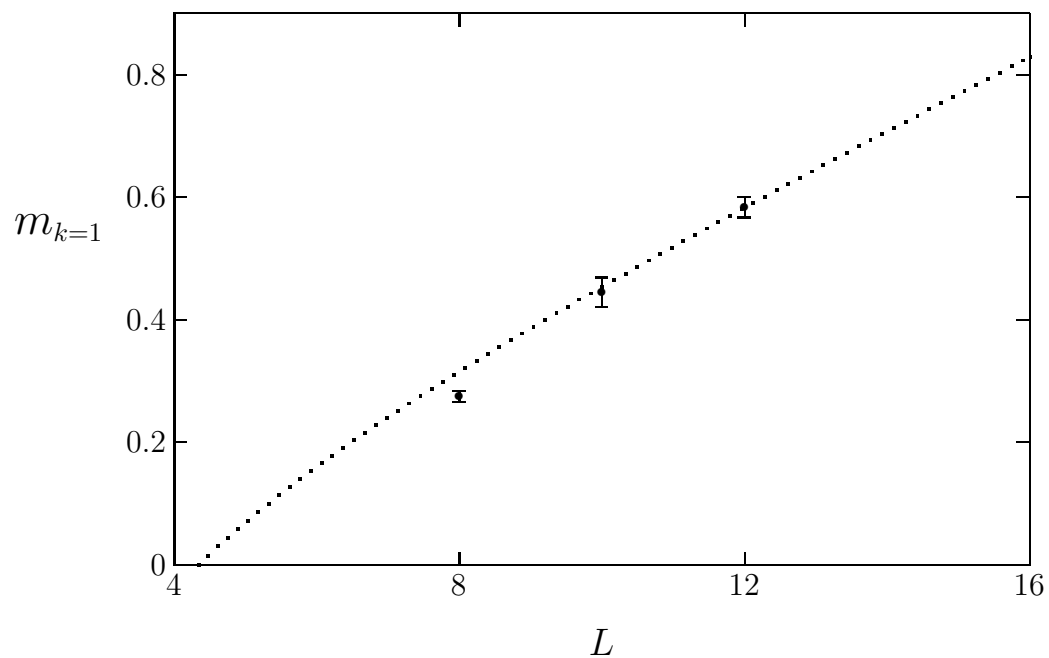


Figure 2: The mass of the fundamental flux loop as a function of its length, at  $\beta = 10.9$  for SU(4). Shown also is the variation one expects from eqn(2).

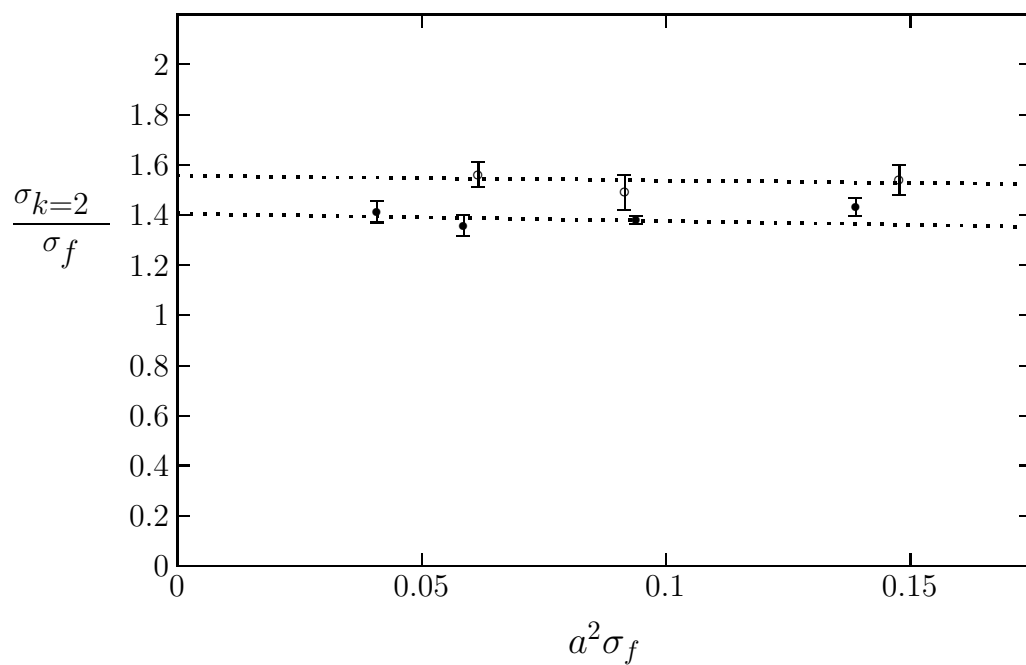


Figure 3: The ratio of  $k = 2$  and  $k = 1$  string tensions in our SU(4)( $\bullet$ ) and SU(5) ( $\circ$ ) lattice calculations plotted as a function of  $a^2\sigma_f$ . Extrapolations to the continuum limit, using a leading  $O(a^2)$  correction, are displayed.